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NON-COOPERATIVE COUNTABLE-PERSON GAMES WITH COMPACT ACTION SPACES

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by

O.J. Vrieze.

ABSTRACT

This paper considers non-cooperative countable-person games. The space of actions for each player is assumed to be compact topological, satisfying the first axiom of countability. The payoff functions of the various players are assumed to be continuous on the product space of the action spaces in the product topology. With the aid of an intrinsic metric introduced on the action spaces it will be shown, that there exists an equilibrium point within the class of mixed actions.

KEY WORDS & PHRASES: *Countable-person games, non-cooperative games, compact topological action spaces, intrinsic metric, equilibrium point.*

1. INTRODUCTION

This paper treats a countable-person non-cooperative game specified by a three-tuple $\Gamma = (I,A,g)$

I: the set of players.

$A = X_{i\in I} A_i$, where A_i, $\forall i \in I$, is the set from which player i will take his actions.

$g = \{g_i | i \in I\}$, where $g_i: A \to \mathbb{R}'$, $\forall i \in I$ and $g_i(a)$ is the payoff to player i if the joint players' actions are $a \in A$.

We make the following assumptions on these game parameters:

A_1: I is a countable set.

A_2: A_i, $\forall i \in I$ is a compact topological space, satisfying the first axiom of countability (cf. KELLEY [5], page 50).

A_3: $g_i(\cdot)$ is a continuous function on A in the product topology and $\sup_{i\in I, a\in A} |g_i(a)| = M$.

Note that as a consequence of Tychonoff's theorem (e.g. ROYDEN [9], page 166) A is compact in the product topology. As I is countable it follows from assumption A_2, that the product topology satisfies the first axiom of countability, so this topology is characterized by sequences (KELLEY [5], theorem 8, page 72). By $A\backslash A_i$ we denote the product space $X_{k\in I, k\neq i} A_k$ and a^{-i} will denote an element of $A\backslash A_i$. Let $G(a) = \Sigma_{i=1}^{\infty} 2^{-i} g_i(a)$, then G(a) is the limit of the sequence $\{\Sigma_{i=1}^{n} 2^{-i} g_i(a)\}$ and as $g_i(a)$ is uniform bounded by M, this sequence of continuous functions converges uniform to G(a), so G(a) is a continuous function on A (see e.g. ROYDEN [9], problem 17, page 149).

Following WALD [11] and TAKAHASHI [10] we now define an intrinsic metric on A_i:

$$(1.1) \qquad \delta^i(a_{i1},a_{i2}) = \max_{a^{-i}\in A\backslash A_i} |G(a^{-i},a_{i1}) - G(a^{-i},a_{i2})|, \forall (a_{i1},a_{i2}) \in A_i \times A_i,$$

where

$$(a^{-i},a_{ik}) = (a_1,\ldots,a_{i-1},a_{ik},a_{i+1},\ldots) \qquad k = 1,2.$$

The space (A_i,δ^i) has now become a pseudo-metric space, which can easily be verified. If $\delta^i(a_{i1},a_{i2}) = 0$, so $g_k(a^{-i},a_{i1}) = g_k(a^{-i},a_{i2})$, $\forall a^{-i}\in A\backslash A_i$, $\forall k \in I$, then it not necessarily holds that $a_{i1} \equiv a_{i2}$. However if $\delta^i(a_{i1},a_{i2}) = 0$ and $\delta^i(a_{i2},a_{i3}) = 0$, then also $\delta^i(a_{i1},a_{i3}) = 0$, so A_i can be partitioned in equivalence classes e_{ij}, in such a way that each two elements of the same class have distance zero. Let E_i be the space formed by these equivalence classes. The metric (1.1) can be extended to E_i as follows: $\delta^i(e_{i1},e_{i2}) = \delta^i(a_{i1},e_{i2})$, $\forall (e_{i1},e_{i2}) \in E_i \times E_i$, where a_{i1} is an arbitrary element of e_{i1} and a_{i2} is an arbitrary element of e_{i2}. From definition (1.1) we see that $\delta^i(e_{i1},e_{i2}) \leq 2M$, $\forall (e_{i1},e_{i2}) \in E_i \times E_i$, $\forall i \in I$.

From the definition (1.1) and the fact that e_{i1} and e_{i2} are equivalence classes, it can easily be seen that it does not matter which $a_{i1} \in e_{i1}$ and $a_{i2} \in e_{i2}$ will be chosen. The space (E_i,δ^i) is a metric space.

When we define $g_k(a_i,\ldots,a_{i-1},e_{ij},a_{i+1},\ldots) = g_k(a_i,\ldots,a_{i-1},a_{i-1},a_{ij},a_{i+1},\ldots)$ where $a_{ij} \in e_{ij}$ arbitrarily, $\forall e_{ij} \in E_i$, $\forall k \in I$, then player i may restrict his pure action set to the set E_i, without drawback on his possibilities to influence his payoff.

We now prove that (E_i,δ^i) is a compact metric space.

If we take a sequence $\{e_{in}\}$ in E_i, then we can correspond with this sequence a sequence $\{a_{in}\}$ in A_i, where $a_{in} \in e_{in}$ arbitrarily. From assumption A_2 it follows that there exist an element $a_{i0} \in A_i$ and a subsequence $\{a_{in'}\}$ of $\{a_{in}\}$ such that $\{a_{in'}\}$ converges to a_{i0} in the topology of A_i. Let e_{i0} be the equivalence class such that $a_{i0} \in e_{i0}$.

Let a_n^{-i} be so that $\delta^i(e_{i0},e_{in}) = \max_{a^{-i}\in A\backslash A_i} \{|G(a^{-i},a_{i0}) - G(a^{-i},a_{in})|\}$

$$(1.2) \qquad = |G(a_n^{-i},a_{i0}) - G(a_n^{-i},a_{in})| \qquad n = 1,2,\ldots$$

This is possible because also $A\backslash A_i$ is a compact topological space satisfying the first axiom of countability and $|G(a^{-i},a_{i0}) - G(a^{-i},a_{in})|$ is as a consequence of assumption A_3 for fixed a_{i0} and a_{in} a continuous function on $A\backslash A_i$. Now there exists an element $a_0^{-i} \in A\backslash A_i$ such that $\{a_{n'}^{-i}\}$ contains a subsequence $\{a_{n''}^{-i}\}$ which converges to a_0^{-i} in the product topology on $A\backslash A_i$. But then the sequence $\{a_{n''}^{-i},a_{in''}\}$ converges to (a_0^{-i},a_{i0}) in the product topology on A.

As G (a) is continuous on A and $A\backslash A_i$, it follows that $G(a^{-i}_{n''},a_{in''}) \to$ $\to G(a^{-i}_0,a_{i0})$ and $G(a^{-i}_{n''},a_0) \to G(a^{-i}_0,a_0)$ as $n'' \to \infty$.
From (1.2) we now see that $\delta^i(e_{i0},e_{in''}) \to 0$ as $n'' \to \infty$. So the arbitrary sequence $\{e_{in}\}$ in E_i contains a convergent subsequence in the metric δ^i and therefore we may conclude that (E_i,δ^i) is a compact metric space.
Of course the above procedure can be carried out for every player.

Let $E = X_{i\in I}E_i$, then as I is countable and as $\delta^i(e_{i1},e_{i2})$, $\forall i$ is uniform bounded, E can be metrized, e.g. $\delta(e_1,e_2) = \Sigma^{\infty}_{i=1}\ 2^{-i}\ \delta^i(e_{i1},e_{i2})$, $\forall(e_1,e_2) \in (E\times E)$, where $e_1 = (e_{11},3_{21},e_{31},\ldots)$ and $e_2 = (3_{12},e_{22},e_{32},\ldots)$.

Define $g_i(\cdot)$ on E as $g_i(e_1) \equiv g_i(a_1)$ where $a_1 = (a_{11},a_{21},a_{31},\ldots)$ with $a_{i1} \in e_{i1}$ arbitrarily. It is easy to see that the choice of a_1 has no influence on this definition as long as $a_{i1} \in e_{i1}$, $\forall i \in I$. We now want to show that $g_i(\cdot)$ is a continuous function on E in the product metric. Let $\{e_n\}$ be a sequence in E which converges to e_0. Assume lim sup $g_i(e_n) \neq g_i(e_0)$. Take $\{a_n\}$ such that $a_n = (a_{1n},a_{2n},\ldots)$ and $a_{in} \in e_{in}$, $\forall i \in I$, $\forall n$. Then there is a subsequence $\{a_{n'}\}$ such that $\lim_{n\to\infty} g_i(a_{n'}) \neq g_i(e_0)$. As the spaces A_i, $\forall i \in I$ satisfy the first axiom of countability we may apply Lemma 30, page 177 of ROYDEN [9] to conclude that the sequence $\{a_{n'}\}$ contains a subsequence $\{a_{n''}\}$ such that the sequence $\{a_{in''}\}$ converges in the topology of A_i for every $i \in I$. Let $a_{i0} = \lim_{n\to\infty} a_{in''}$, $\forall i \in I$. Then

$$g_k(a^{-i},a_{in''}) \to g_k(a^{-i},a_{i0}), \qquad \forall k \in I,\ \forall a^{-i} \in A\backslash A_i,$$

which means $\delta^i(e_{in''},e_{i*}) \to 0$, where e_{i*} is the element of E_i to which a_{i0} belongs, $\forall i \in I$.
But then $\delta(e_{n''},e_*) \to 0$ and in combination with $\{e_n\} \to e_0$, this yields $e_* \equiv e_0$.

Now $g_i(e_0) = g_i(e_*) = g_i(a_0) = \lim_{n\to\infty} g_i(a_{n''}) = \lim_{n\to\infty} g_i(a_{n'}) \neq g_i(e_0)$ so starting from the assumption lim sup $g_i(e_n) \neq g_i(e_0)$, we have come to a contradiction and therefore lim sup $g_i(e_n) = g_i(e_0)$ and in the same way we can show lim inf $g_i(e_n) = g_i(e_0)$ so that lim $g_i(e_n) = g_i(e_0)$, or $g_i(\cdot)$ is continuous on E.

In section 2 we prove that the game $\Gamma = (I,E,g)$ possesses an equilibrium point within the class of mixed actions and in section 3 we show that this equilibrium point also represents an equilibrium point in the original game.

Let m_i be the σ-algebra of Borelsubsets of (E_i,δ^i) and let N_i denote the set of all probability measures on E_i defined for each $H \in M_i$. N_i must be seen as the mixed action space of player i, i.e. if player i decides to play a mixed action $\mu_i \in N_i$ then with the aid of a chance mechanisme according to μ_i, he selects a pure $e_i \in E_i$. On N_i we define the weak topology (see e.g. PARTASARATHY [7] page 39). As E_i is compact metric, it follows from PARTHASARATHY [7] (theorem 6.4, page 45), that N_i is compact and can be metrized, so the weak topology of N_i satisfies the first axiom of countability (KELLEY [5], theorem 17, page 125). Let $N = X_{i\in I} N_i$, endowed with the product topology, be the space of all product probability measures on E, defined on the product σ-field in E. Note that as I is countable, also the product topology on N satisfies the first axiom of countability, so the topology on N is characterized by sequences (see e.g. KELLEY [5] theorem 8, page 72). An element of N will be denoted by $\mu = (\mu_1,\mu_2,\ldots)$ and if the game is played with player i playing $\mu_i \in N_i$, $\forall i \in I$, then the expected payoff to player i can be written as:

$$g_i(\mu) = \int_E g_i(e)d\mu(e) \qquad \text{where } \mu = (\mu_i,\mu_2,\ldots).$$

Definition: an element $\mu^* = (\mu_i^*,\mu_2^*,\ldots) \in N$ is called an equilibrium point for the game $\Gamma = (I,E,g)$ iff:

$$g_i(\mu^{-i^*},\mu_i) \leq g_i(\mu^*), \qquad \forall \mu_i \in N_i,\ \forall i \in I$$

where

$$(\mu^{-i^*},\mu_i) = (\mu_1^*,\ldots,\mu_{i-1}^*,\mu_i,\mu_{i+1}^*,\ldots).$$

When we make the simplification, that the set of players is finite, then we get a model which is earlier studied by GLICKSBERG [3] and NIKAIDÔ-ISODA [6]. Glicksberg showed the existence of an equilibrium point under nearly the same assumptions as we made. He used a point to convex set mapping which appeared to be upper semi-continuous in the mixed strategy space and showed in his paper that if the mixed strategy space is linear Hausdorff topological, then this mapping must have a fixed point, which proved to be an equilibrium point.

Nikaidô and Isoda have treated convex games. By a convex game they mean a N-person non-cooperative game under the following assumptions:

a) Player i^{th} strategy space is a compact convex subset A_i of a topological linear space.
b) Player i^{th} payoff function $g_i(a_1,\ldots,a_i,\ldots,a_N)$ is concave with respect to his own strategy variable $a_i \in A_i$.
c) The sum of the payoffs $\Sigma_{i=1}^{N} g_i(\cdot)$ is continuous over $A = X_{i=1}^{N} A_i$.
d) For each a_i $g_i(a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_N)$ is a continuous function of the (N-1)-tuple $(a_i,\ldots,a_{i-1},a_{i+1},\ldots,a_N) \in A\backslash A_i$.

Under these assumptions they proved the existence of an equilibrium point within the class of pure actions A, with the aid of some convexity theory. Note that in our model N_i the set of mixed actions is a compact convex set, $g_i(\mu^{-i},\mu_i)$ is linear in μ_i, so also concave and in section 2 we prove that $g_i(\cdot)$ is continuous on $N = X_{i\in I}N_i$ from which we see that also the assumptions c) and d) are fulfilled.
So when we take I to be finite, then Glicksberg and also Nikaidô-Isoda have solved our problem.
With the aid of our Lemma 2.2 the method of Glicksberg can be extended to the case where I is countable.
It is unclear to us if also the method of Nikaidô-Isoda can be extended to our case.
However we present a new method of solving the problem. The main reason for this is, that our method proves to be very useful for attacking stochastic games which will be shown in a later paper.
By □ we denote the end of a proof.

2. EXISTENCE OF AN EQUILIBRIUM POINT

Denote by F_i the set of all finite signed measures on E_i, defined for each $H \in M_i$, endowed with the weak topology and let $F = X_{i\in I} F_i$ be the set of all finite signed product measures on E defined on the product σ-field of E, endowed with the product topology. The space $N = X_{i\in I} N_i$ is a real subset of F.

<u>LEMMA 2.1</u>. F_i *is a linear Hausdorff topological space and therefore also* $F = X_{i\in I} F_i$ *is linear Hausdorff in the product topology.*

<u>PROOF</u>. If $\mu_i, \nu_i \in F_i$ and $\alpha,\beta \in \mathbb{R}'$ and define ρ_i as $\rho_i(H) = \alpha\mu_i(H) + \beta\nu_i(H)$, $\forall H \in M_i$, then it is easy to see that $\rho_i \in F_i$, so F_i is linear.
Let $\mu_i^0, \nu_i^0 \in F_i$ and $\mu_i^0 \neq \nu_i^0$, then it follows from PARTHASARATHY [7] (theorem 5.9 page 39) that there exists a bounded real-valued uniform continuous function $f_i(\cdot)$ on E_i, so that:

$$\int_{E_i} f_i(e_i)\, d\mu_i^0(e_i) \neq \int_{E_i} f_i(e_i)\, d\nu_i^0(e_i)$$

Choose $\varepsilon > 0$, so that

$$\left|\int_{E_i} f_i(e_i)\, d\mu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i^0(e_i)\right| > \varepsilon$$

Let

$$O_{\mu_i^0} = \left\{\mu_i \;\middle|\; \mu_i \in F_i \text{ and } \left|\int_{E_i} f_i(e_i)\, d\mu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i(e_i)\right| < \frac{\varepsilon}{2}\right\}$$

be an open neighbourhood of μ_i^0; of course $\nu_i^0 \notin O_{\mu_i^0}$.
Let

$$O_{\nu_i^0} = \left\{\nu_i \;\middle|\; \nu_i \in F_i \text{ and } \left|\int_{E_i} f_i(e_i)\, d\nu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\nu_i(e_i)\right| < \frac{\varepsilon}{2}\right\}$$

be an open neighbourhood of ν_i^0; of course $\mu_i^0 \notin O_{\nu_i^0}$.

Choose $\mu_i \in \mathcal{O}_{\mu_i^0}$ arbitrarily, then

$$\left|\int_{E_i} f_i(e_i)\, d\nu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i(e_i)\right| =$$

$$\left|\int_{E_i} f_i(e_i)\, d\nu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i^0(e_i) + \int_{E_i} f_i(e_i)\, d\mu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i(e_i)\right| \geq$$

$$\geq \left|\int_{E_i} f_i(e_i)\, d\nu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i^0(e_i)\right| - \left|\int_{E_i} f_i(e_i)\, d\mu_i^0(e_i) - \int_{E_i} f_i(e_i)\, d\mu_i^0(e_i)\right| \geq$$

$$\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

so $\mu_i \notin \mathcal{O}_{\nu_i^0}$.

In the same way $\nu_i \in \mathcal{O}_{\nu_i^0} \Rightarrow \nu_i \notin \mathcal{O}_{\mu_i^0}$.

So $\mathcal{O}_{\mu_i^0}$ and $\mathcal{O}_{\nu_i^0}$ are disjunct open neighbourhoods of μ_i^0 and ν_i^0 respectively and so F_i is Hausdorff. □

<u>LEMMA 2.2</u>. $g_i(\mu)$ *is a continuous function on* N, $\forall i \in I$.

<u>PROOF</u>. The proof is an extension of the proof of lemma 2.1 in PARTHASARATHY & MAITRA [8] as used in FEDERGRÜN [2]. First note that if $\mu_n \to \mu_0$ in the product topology on N, then $\mu_{in} \to \mu_{i0}$, $\forall i \in I$ on N_i in the weak topology.

Consider now the family G(E) of continuous real-valued functions on E of the form $\Sigma_{j=1}^{K}\ \pi_{i\in I}\ f_{ij}(\cdot)$ where $f_{ij}(\cdot)$ is a continuous real-valued function on E_i, $\forall i \in I$ and $f_{ij}(\cdot) \equiv 1$ for all but a finite number. Then it is clear that this family G(E) is an algebra (closed under finite linear combinations and finite products). Also this family contains the constant functions ans separates the points of E. To see this last assertion, choose $e_1, e_2, \in E$, $e_1 \neq e_2$, then there is at least one coordinate j such that $e_{j1} \neq$ $\neq e_{j2}$. Let $f_j^*(\cdot)$ be a continuous real-valued function on E_j, such that $f_j^*(e_{j1}) \neq f_j^*(e_{j2})$ (This is possible because of Urysohn's lemma (ROYDEN [9], page 148)). Then the function $\pi_{i\in I}\ f_i(\cdot) \in G(E)$ defined as $f_i(\cdot) \equiv 1$, $i \neq j$ and $f_j(\cdot) \equiv f_j^*(\cdot)$, separates the points e_1 and e_2.

So all the conditions necessary for applying the Stone-Weierstrass theorem (ROYDEN [9], page 174) are fulfilled.

This means that, since $g_i(\cdot)$ is a continuous function on E, this function can be uniform approximated by a sequence out of G(E). So for every $\varepsilon > 0$ we can find a member $\Sigma_{j=1}^{k}\ \pi_{i\in I}\ f_{ij}^*(\cdot)$ of G(E) such that

$$(2.1) \qquad \left|\sum_{j=1}^{k}\ \pi_{i\in I}\ f_{ij}^*(e_i) - g_i(e)\right| < \frac{\varepsilon}{4}, \qquad \forall e \in E.$$

Now consider the expression

$$\int_E\ \pi_{i\in I}\ f_{ij}(e_i)\ d\mu(e), \qquad u \in N.$$

As only a finite number, say m, of the $f_{ij}(\cdot) \not\equiv 1$, we can rearrange the coordinates to get

$$(2.2) \qquad \int_E\ \pi_{i\in I}\ f_{ij}(e_i)\ d\mu(e) = \int_E\ \pi_{i=1}^{m}\ f_{ij}(e_i)\ d\mu(e).$$

Let $\{\phi_{in}(e_i)\}$ be a sequence of simple functions on E_i, such that

$$\lim_{n\to\infty} \phi_{in}(e_i) = f_{ij}(e_i), \qquad \forall e_i \in E_i,\ i = 1,\ldots,m.$$

Then

$$\lim_{n\to\infty}\ \pi_{i=1}^{m}\ \phi_{in}(e_i) \to \pi_{i=1}^{m}\ f_{ij}(e_i), \qquad \forall e \in N$$

(all functions $f_{ij}(\cdot)$ are uniform bounded) and $\pi_{i=1}^{m}\ \phi_{in}(e_i)$ is a simple function on E.

So

$$(2.3) \qquad \int_E \prod_{i=1}^{m} f_{ij}(e_i)\, d\mu(e) = \lim_{n\to\infty} \int_E \prod_{i=1}^{m} \phi_{in}(e_i)\, d\mu(e).$$

Now

$$(2.4) \qquad \int_E \prod_{i=1}^{m} \phi_{in}(e_i)\, d\mu(e) = \sum_{j_1=1}^{n(1)} \cdots \sum_{j_m=1}^{n(m)} \left\{ \prod_{i=1}^{m} \phi_{in}(j_i) \right\} \mu(E_{1j_1},\ldots, E_{mj_m}, E_{m+1},\ldots)$$

where $n(i)$ is the finite number of various values which the function $\phi_{in}(\cdot)$ takes on, $i = 1,\ldots,m$ and E_{ij_i} is the Borel measurable subset of E_i where $\phi_{in}(\cdot)$ has constant value $\phi_{in}(j_i)$, $i = 1,\ldots,m$. As $\mu(\cdot)$ is a probability measure defined on the product σ-algebra of E, we see from HALMOS [4] (Theorem B, page 157) that

$$(2.5) \qquad \mu(E_{1j_1},\ldots,E_{mj_m},E_{m+1},\ldots) = (\mu_1\times\mu_2\times\ldots\times\mu_m)(E_{1j_1},\ldots,E_{mj_m})$$

and

$$(2.6) \qquad (\mu_1\times\mu_2\times\ldots\times\mu_m)(E_{1j_1},\ldots,E_{mj_m}) = \prod_{i=1}^{m} \mu_i(E_{ij_i})$$

as a consequence of Fubini's theorem (e.g. HALMOS [4]; theorem C, page 148). Combining (2.2), (2.3), (2.4), (2.5) and (2.6) yields:

$$\int_E \prod_{i\in I} f_{ij}(e_i)\, d\mu(e) = \int_E \prod_{i=1}^{m} f_{ij}(e_i)\, d\mu(e) = \lim_{n\to\infty} \int_E \prod_{i=1}^{m} \phi_{in}(e_i)\, d\mu(e) =$$

$$= \lim_{n\to\infty} \sum_{j_1=1}^{n(1)} \cdots \sum_{j_m=1}^{n(m)} \left\{ \prod_{i=1}^{m} \phi_{in}(\gamma_i) \right\} \mu(E_{1j_1},\ldots,E_{mj_m},E_{m+1},\ldots) =$$

$$= \lim_{n\to\infty} \sum_{j_1=1}^{n(1)} \cdots \sum_{j_m=1}^{n(m)} \prod_{i=1}^{m} \phi_{in}(j_i)\, \mu_i(E_{ij_i}) =$$

$$= \prod_{i=1}^{m} \lim_{n\to\infty} \sum_{j_i=1}^{n(i)} \phi_{in}(\gamma_i)\, \mu_i(E_{ij_i}) = \prod_{i=1}^{m} \int_{E_i} f_{ij}(e_i)\, d\mu_i(e_i).$$

So

$$(2.7) \qquad \int_E \prod_{i\in I} f_{ij}(e_i)\, d\mu(e) = \prod_{i=1}^{m} \int_{E_i} f_{ij}(e_i)\, d\mu_i(e_i), \qquad \forall\mu \in N$$

if $f_{ij}(\cdot)$, $i = 1,\dots,m$ are the only functions which are not the constant function 1. So if $\Sigma_{j=1}^{n} \pi_{i\in I} f_{ij}(e_i) \in G(E)$ then from (2.7):

$$(2.8) \qquad \int_E \sum_{j=1}^{k} \prod_{i\in I} f_{ij}(e_i)\, d\mu(e) = \sum_{j=1}^{k} \prod_{i\in I} \int_{E_i} f_{ij}(e_i)\, d\mu_i(e_i), \qquad \forall\mu \in N,$$

whereby only a finite number of expressions $\int_{E_i} f_{ij}(e_i)\, d\mu_i(e_i)$ unequals 1.

Now if $\mu_n \to \mu_0$ in N, so $\mu_{in} \to \mu_{i0}$, $\forall i \in I$, then because of (2.8) and $\int_{E_i} f_{ij}(e_i)\, d\mu_{in}(e_i) \to \int_{E_i} f_{ij}(e_i)\, d\mu_{i0}(e_i)$ and the fact that the right-hand side of (2.8) depends only on a finite number of coordinates μ_i, we may conclude that

$$(2.9) \qquad \int_E \sum_{j=1}^{k} \prod_{i\in I} f_{ij}(e_i)\, d\mu_n(e) \to \int_E \sum_{j=1}^{k} \prod_{i\in I} f_{ij}(e_i)\, d\mu_0(e),$$

$$\forall \sum_{j=1}^{k} \prod_{i\in I} f_{ij}(e_i) \in G(E).$$

Now

$$\left| \int_E g_i(e)\, d\mu_n(e) - \int_E g_i(e)\, d\mu_0(e) \right| \le$$

$$\le \left| \int_E \sum_{j=1}^{k} \prod_{i\in I} f^*_{ij}(e_i)\, d\mu_n(e) - \int_E \sum_{j=1}^{k} \prod_{i\in I} f^*_{ij}(e_i)\, d\mu_0(e) \right| + \frac{\varepsilon}{2} \le$$

(see (2.1))

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{as } n > N(\varepsilon).$$

This last step is possible because of (2.9) □

We now introduce a mapping T: $N \to N$ which possesses two properties, by which we will be able to show the existence of an equilibrium point.

Let $\mu \in N$ and let $g_i(\mu)$ the expected payoff to player i under this joint mixed action. Let

$$(2.10) \qquad y_i(\mu,e_i) \equiv g_i(\mu^{-i},e_i), \qquad \forall e_i \in E_i, \forall i \in I,$$

where $g_i(\mu^{i},e_i)$ represents the expected payoff to player i, if he takes action e_i and the other players play according to μ. Let

$$(2.11) \qquad \phi_i(\mu,e_i) \equiv \max\{0,y_i(\mu,e_i) - g_i(\mu)\}, \qquad \forall e_i \in E_i, \forall i \in I.$$

In section 1 we showed that (E_i,δ^i) was a compact metric space, so there exist a countable subset $E_i^* = \{e_{in}^*, n = 1,2,\ldots\}$ of E_i which is dense in E_i. Let $\lambda_i \in N_i$ be the probability measure defined as

$$\lambda_i(e_{in}^*) = 2^{-n}, \qquad n = 1,2,\ldots, \text{ so } \lambda_i(E_i^*) = 1.$$

Let

$$(2.12) \qquad \Phi_i(\mu,H) \equiv \int_H \phi_i(\mu,e_i)\, d\lambda_i(e_i) = \sum_{e_{in}^*\in H} \lambda_i(e_{in}^*)\, \phi_i(\mu,e_{in}^*), \qquad \forall H \in m_i, \forall i \in I.$$

Note that $\Phi_i(\mu,\cdot)$ is a measure on E_i concentrated on E_i^*.
Define now the mapping $T: N \to N$ as:

$$(2.13) \qquad (T\mu)_i\,(H) = \frac{\mu_i(H)+\Phi_i(\mu,H)}{1+\Phi_i(\mu,E_i)} \qquad \forall H \in M_i, \forall i \in I.$$

It is easy to see that $(T\mu)_i \in N_i$, $\forall i \in I$, so $T\mu \in N$.

We prove now two properties of this mapping T.

Property 1: $T: N \to N$ is a continuous mapping and possesses a fixed point.

Property 2: Every fixed point of T is an equilibrium point for the game $\Gamma = (I,E,g)$ and conversely.

PROOF OF PROPERTY 1: Lemma 2.2 tells us that $y_i(\mu,e_i)$ is continuous in μ

for fixed e_i and continuous in e_i for fixed μ. For arbitrary $H \in M_i$ we can order the set of points $H \cap E_i^*$ according to decreasing value of $\lambda_i(e_{in}^*)$, $e_{in}^* \in H \cap E_i^*$. Let $\{e_{in'}^*\}$, $n' = 1,2,\ldots,k(H)$ be this sequence where $k(H)$ may be ∞. If H is such that $k(H)$ is finite, then $\Phi_i(\mu,H)$ is a weighted combination of a finite number of continuous functions in μ and therefore $\Phi_i(\mu,H)$ itself is also continuous in μ for this H. If H^0 is such that $k(H^0) = \infty$, then $\lim_{n'\to\infty} \lambda^i(e_{in'}^*) = 0$.

Since $|g_i(\mu)|$ is uniform bounded by M, it follows from (2.10) and (2.11) that

$$|\phi_i(\mu,e_i)| \le 2M, \qquad \forall e_i \in E_i, \forall \mu \in N.$$

Choose N', such that $\lambda_i(e_{iN'}^*) \le \varepsilon/16M$, for fixed $\varepsilon > 0$. Then for H^0:

$$(2.14) \qquad |\Phi_i(\mu,H^0) - \sum_{n'=1}^{N'} \lambda_i(e_{in'}^*)\ \phi_i(\mu,e_{in'}^*)| \le \frac{\varepsilon}{4}, \qquad \forall \mu \in N.$$

If $\mu_n \to \mu_0$ then there exists a $N(\varepsilon)$ such that if $n > N(\varepsilon)$, then

$$(2.15) \qquad |\sum_{n'=1}^{N'} \lambda_i(e_{in'}^*)\ \phi_i(\mu_n,e_{in'}^*) - \sum_{n'=1}^{N'} \lambda_i(e_{in'}^*)\ \phi_i(\mu_0,e_{in'}^*)| \le \frac{\varepsilon}{2}.$$

Combining (2.14) and (2.15) gives

$$|\Phi_i(\mu_n,H^0) - \Phi_i(\mu_0,H^0)| < \varepsilon, \ \forall n > N,$$

so $\Phi_i(\mu,H)$ is a continuous function in μ for every $H \in M_i$. Especially $\Phi_i(\mu,E_i)$ is continuous in μ. Let

$$\nu_i(\mu,H) \equiv \frac{\mu_i(H)}{1+\Phi_i(\mu,E_i)} \text{ and } \rho_i(\mu,H) = \frac{\Phi_i(\mu,H)}{1+\Phi_i(\mu,E_i)}, \qquad \forall H \in M_i,$$

then $\qquad (T\mu)_i = \nu_i(\mu,H) + \rho_i(\mu,H).$

Then we see that $\nu_i(\mu,\cdot)$ is a measure on E_i which is weakly continuous in μ, i.e. if $\mu_n \to \mu_0$ in the product topology on N, then $\nu_i(\mu_n,\cdot) \to \nu_i(\mu_0,\cdot)$ in the weak topology on N_i.

$\rho_i(\mu,\cdot)$ is a measure on E_i which is setwise continuous on N, i.e. if $\mu_n \to \mu_0$ in the topology on N, then $\rho_i(\mu_n,H) \to \rho_i(\mu_0,H)$, $\forall H \in M_i$. As a setwise continuous measure is also weakly continuous, we may conclude that $(T\mu)_i$ is also weakly continuous in $\mu \in N$, $\forall i \in I$. So the mapping $T: N \to N$ is continuous in the product topology.

As a consequence of the Schauder-Tychonoff theorem (e.g. DUNFORD & SCHWARZ [1], page 456), which says that a continuous mapping $T: N \to N$, where N is a convex compact subset of a linear Hausdorff topological space F, possesses at least one fixed point, we now can conclude that the mapping T, as defined in (2.13), possesses a fixed point. □

PROOF OF PROPERTY 2:

a) Let $\mu^* \in N$ be equilibrium point, then by definition of equilibrium point:

$$y_i(\mu^*,e_i) = g_i(\mu^{-i^*},e_i) \le g_i(\mu^*) \qquad \forall e_i \in E_i,\ \forall i \in I. \tag{2.16}$$

$$(2.16) \text{ and } (2.11) \Rightarrow \phi_i(\mu^*,e_i) = 0, \quad \forall e_i \in E_i,\ \forall i \in I \tag{2.17}$$

$$(2.17) \text{ and } (2.12) \Rightarrow \Phi_i(\mu^*,H) = 0, \quad \forall H \in M_i,\ \forall i \in I \tag{2.18}$$

$$(2.18) \text{ and } (2.13) \Rightarrow (T\mu^*)_i = \mu_i^* \qquad \forall i \in I,$$

so μ^* is a fixed point of T.

b) Let μ^* be a fixed point of T. From (2.13) we see that μ^* satisfies:

$$\mu_i^*(H) \cdot \Phi_i(\mu^*,E_i) = \Phi_i(\mu^*,H), \qquad \forall H \in M_i,\ \forall i \in I. \tag{2.19}$$

We now assume that $\Phi_i(\mu^*,E_i) > 0$ and show that this leads to a contradiction. If $\Phi_i(\mu^*,E_i) > 0$, then from (2.19) we see that

$$\mu_i^*(H) = \frac{\Phi_i(\mu^*,H)}{\Phi_i(\mu^*,E_i)}. \tag{2.20}$$

Since $\Phi_i(\mu^*,\cdot)$ is a measure on E_i concentrated on E_i^*, it can be seen from (2.20) that $\mu_i^*(\cdot)$ is a probability measure on E_i concentrated on E_i^* and

$$(2.21) \qquad \mu_i^*(e_i) = \frac{\phi_i(\mu^*,e_i)\cdot\lambda_i(e_i)}{\Phi_i(\mu^*,E_i)} .$$

From (2.21) we see that $\mu_i^*(e_i) > 0$ if and only if $\phi_i(\mu^*,e_i) > 0$ and $e_i \in E_i^*$.
From the assumption $\Phi_i(\mu^*,E_i) > 0$ it follows that there is at least one $e_i \in E_i^*$ with $\phi_i(\mu^*,e_i) > 0$. Now we see that

$$g_i(\mu^*) = \int_{E_i} g_i(\mu^{-i^*},e_i)\, d\mu_i^*(e_i) = \int_{E_i} y_i(\mu^*,e_i)\, d\mu_i^*(e_i) =$$

$$= \sum_{\substack{e_i\in E_i^* \\ \mu_i^*(e_i)>0}} y_i(\mu^*,e_i)\, \mu_i^*(e_i) > \sum_{\substack{e_i\in E_i^* \\ \mu_i^*(e_i)>0}} g_i(\mu^*)\, \mu_i^*(e_i) = g_i(\mu^*).$$

So we encounter a contradiction and therefore our assumption $\Phi_i(\mu^*,E_i) >$ > 0 appears to be false.
So we may conclude

$$(2.22) \qquad \Phi_i(\mu^*,E_i) = 0.$$

From (2.10), (2.11), (2.12) and (2.22) it follows that

$$(2.23) \qquad g_i(\mu^{-i^*},e_i) \le g_i(\mu^*), \qquad \forall e_i \in E_i^*,\ \forall i \in I.$$

Then (2.23) together with the continuity of $g_i(\mu^{-i^*},e_i)$ in e_i and the fact that E_i^* is a dense subset of E_i enables us to conclude

$$(2.24) \qquad g_i(\mu^{-i^*},e_i) \le g_i(\mu^*), \qquad \forall e_i \in E_i,\ \forall i \in I.$$

As

$$g_i(\mu^{-i^*},\mu_i) \le \max_{e_i\in E_i} g_i(\mu^{-i^*},e_i), \qquad \forall \mu_i \in N_i,$$

we have the desired result $g_i(\mu^{-i^*},\mu_i) \le g_i(\mu^*)$: $\forall \mu_i \in N_i$, $\forall i \in I$. □

THEOREM 2.1. *The game* $\Gamma = (I,E,g)$ *possesses an equilibrium point.*

PROOF. Combining the two properties of the mapping T as defined in (2.13) gives the desired result. □

3. RETURN TO THE ORIGINAL PROBLEM

We denote the original topology of A_i by T_i. In section 1 we defined a function G(a) which is continuous on $A = X_{i\in I} A_i$ in the product topology and $|G(a)| \le M$, $\forall a \in A$. As $A\backslash A_i$ is compact it is easy to see that the function

$$\max_{a^{-i}\in A\backslash A_i} |G(a^{-i},a_i) - G(a^{-i},a_{i0})|$$

is continuous on A_i for fixed a_{i0} in the topology T_i.

Let Θ_i denote the set of open subsets of the metric space (E_i,δ^i). It is well-known that for a metric space a base at a point of this space for the topology Θ_i is the countable set of open spheres of rational radius at this point. So

$$\{\mathcal{O}_{e_{i0}}(r) \mid \mathcal{O}_{e_{i0}}(r) = \{e_i \mid e_i \in E_i;\ \delta^i(e_{i0},e_i) < r\},\ r \text{ rational}\}$$

is a base at a point $e_{i0} \in E_i$ for the topology Θ_i.

Remember that an element $e_i \in E_i$ can be seen as a subset of A_i. Then from the above with $a_{i0} \in e_{i0}$ arbitrarily:

$$\mathcal{O}_{e_{i0}}(r) = \{e_i \mid \delta^i(e_i,e_{i0}) < r\} =$$

$$= \bigcup_{e_i\in\mathcal{O}_{e_{i0}}(r)} \{a_i \mid a_i \in A_i;\ a_i \in e_i\} =$$

$$= \{a_i \mid \delta^i(a_i,a_{i0}) < r\} =$$

$$= \{a_i \mid \max_{a^{-i}\in A\backslash A_i} |G(a^{-i},a_i) - G(a^{-i},a_{i0})| < r\} \in T_i.$$

But this means that there is a base for Θ_i which is a subset of the topology T_i.

As each element of the topology Θ_i is the union of members of his base we see that $\Theta_i \subset T_i$. So the metric topology of E_i is weaker than the original topology T_i. Then the σ-algebra of Borel subsets of (A_i, T_i) contains the σ-algebra of Borel subsets of (E_i, δ^i).

In section 1 we defined N_i as the set of probability measures defined on the σ-algebra M_i of Borel subsets of (E_i, δ^i). Let N_i^* be the set of probability measures defined on the σ-algebra M_i^* of Borel subsets of (A_i, T_i). From the above it may be clear that $N_i \subset N_i^*$. So for the equilibrium point $\mu^* \in N$ in section 2 it holds that $\mu_i^* \in N_i^*$, $\forall i \in I$ and as

$$g_i(\mu^{-i^*}, e_i) \le g_i(\mu^*), \qquad \forall e_i \in E_i$$

it also holds that

$$g_i(\mu^{-i^*}, a_i) \le g_i(\mu^*), \qquad \forall a_i \in A_i.$$

This last inequality ensures that

$$g_i(\mu^{-i^*}, \mu_i) \le g_i(\mu^*) \quad \forall \mu_i \in N_i^*, \qquad \forall i \in I.$$

But this means that μ^* is also equilibrium point in the original class of mixed actions $N^* = X_{i \in I} N_i^*$.

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We would like to thank Arie Hordijk and Gerard Wanrooij for tempering too quick observations and for many helpful comments.

In a later paper we will extend the method of section 2 to the case of stochastic games, where an appropriate mapping appears to possess the same properties as the mapping T in section 2.

REFERENCES

[1] DUNFORD, N. & J.T. SCHWARTZ, *Linear Operations, part I: General Theory*, Interscience Publishers Inc., New York, 1967.

[2] FEDERGRÜN, A., in cooperation with O.J. VRIEZE & G.L. WANROOIJ, *On the existence of discounted and average return equilibrium policies in* N*-person games*, Report BW 57/75, Mathematisch Centrum, Amsterdam, 1975.

[3] GLICKSBERG, I.L., *A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points*, Proc. Amer. Math. Soc.3, 1952, (170-174).

[4] HALMOS, P.R., *Measure Theory*, Springer Verlag, New York, 1950.

[5] KELLEY, J.L., *General Topology*, van Nostrand, New York, 1961.

[6] NIKAIDÔ, H. & K. ISODA, *Note on non-cooperative convex games*, Pacific J. Math.5, 1955, (807-815).

[7] PARTHASARATHY, K.R., *Probability measures on metric spaces*, Academic Press, New York-London, 1967.

[8] PARTHASARATHY, T. & A. MAITRA, *On Stochastic Games*, Journal of Opt. Theory and Appl.5, 1970, (209-300).

[9] ROYDEN, H.L., *Real Analysis*, sec. ed. The McMillan Company, New York, 1968.

[10] TAKAHASHI, M., *Stochastic games with infinitely many strategies*, J. Sci. Hiroshima Univ. Serie A-I: 26, 1962, (123-134).

[11] WALD, A., *Statistical Decision Functions*, John Wiley & Sons, New York, 1950.